

ON PREPERIODIC POINTS OF RATIONAL FUNCTIONS DEFINED OVER $\mathbb{F}_p(t)$

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ABSTRACT. Let $P \in \mathbb{P}_1(\mathbb{Q})$ be a periodic point for a monic polynomial with coefficients in \mathbb{Z} . With elementary techniques one sees that the minimal periodicity of P is at most 2. Recently we proved a generalization of this fact to the set of all rational functions defined over \mathbb{Q} with good reduction everywhere (i.e. at any finite place of \mathbb{Q}). The set of monic polynomials with coefficients in \mathbb{Z} can be characterized, up to conjugation by elements in $\mathrm{PGL}_2(\mathbb{Z})$, as the set of all rational functions defined over \mathbb{Q} with a totally ramified fixed point in \mathbb{Q} and with good reduction everywhere. Let p be a prime number and let \mathbb{F}_p be the field with p elements. In the present paper we consider rational functions defined over the rational global function field $\mathbb{F}_p(t)$ with good reduction at every finite place. We prove some bounds for the cardinality of orbits in $\mathbb{F}_p(t) \cup \{\infty\}$ for periodic and preperiodic points..

Keywords. preperiodic points, function fields.

1. INTRODUCTION

In arithmetic dynamic there is a great interest about periodic and preperiodic points of a rational function $\phi: \mathbb{P}_1 \rightarrow \mathbb{P}_1$. A point P is said to be *periodic* for ϕ if there exists an integer $n > 0$ such that $\phi^n(P) = P$. The minimal number n with the above properties is called *minimal* or *primitive period*. We say that P is a *preperiodic point* for ϕ if its (forward) orbit $O_\phi(P) = \{\phi^n(P) \mid n \in \mathbb{N}\}$ contains a periodic point. In other words P is preperiodic if its orbit $O_\phi(P)$ is finite. In this context an orbit is also called a cycle and its size is called the length of the cycle.

Let p be a prime and, as usual, let \mathbb{F}_p be the field with p elements. We denote by K a global field, i. e. a finite extension of the field of rational numbers \mathbb{Q} or a finite extension of the field $\mathbb{F}_p(t)$. Let D be the degree of K over the base field (respectively \mathbb{Q} in characteristic 0 and $\mathbb{F}_p(t)$ in positive characteristic). We denote by $\mathrm{PrePer}(\phi, K)$ the set of K -rational preperiodic points for ϕ . By considering the notion of height, one sees that the set $\mathrm{PrePer}(\phi, K)$ is finite for any rational map $\phi: \mathbb{P}_1 \rightarrow \mathbb{P}_1$ defined over K (see for example [13] or [5]). The finiteness of the set $\mathrm{PrePer}(f, K)$ follows from [5, Theorem B.2.5, p.179] and [5, Theorem B.2.3, p.177] (even if these last theorems are formulated in the case of number fields, they have a similar statement in the function field case). Anyway, the bound deduced by those results depends strictly on the coefficients of the map ϕ (see also [13, Exercise 3.26 p.99]). So, during the last twenty years, many dynamists have searched for bounds that do not depend on the coefficients of ϕ . In 1994 Morton and Silverman stated a conjecture known with the name "Uniform Boundedness Conjecture for Dynamical Systems": for any number field K , the number of K -preperiodic points

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of a morphism $\phi: \mathbb{P}_N \rightarrow \mathbb{P}_N$ of degree $d \geq 2$, defined over K , is bounded by a number depending only on the integers d, N and $D = [K : \mathbb{Q}]$. This conjecture is really interesting even for possible application on torsion points of abelian varieties. In fact, by considering the Lattès map associated to the multiplication by two map [2] over an elliptic curve E , one sees that the Uniform Boundedness Conjecture for $N = 1$ and $d = 4$ implies Merel's Theorem on torsion points of elliptic curves (see [6]). The Lattès map has degree 4 and its preperiodic points are in one-to-one correspondence with the torsion points of $E/\{\pm 1\}$ (see [11]). So a proof of the conjecture for every N , could provide an analogous of Merel's Theorem for all abelian varieties. Anyway, it seems very hard to solve this conjecture, even for $N = 1$.

Let R be the ring of algebraic integers of K . Roughly speaking: we say that an endomorphism ϕ of \mathbb{P}_1 has (simple) good reduction at a *place* \mathfrak{p} if ϕ can be written in the form $\phi([x : y]) = [F(x, y), G(x, y)]$, where $F(x, y)$ and $G(x, y)$ are homogeneous polynomial of the same degree with coefficients in the local ring $R_{\mathfrak{p}}$ at \mathfrak{p} and such that their resultant $\text{Res}(F, G)$ is a \mathfrak{p} -unit. In Section 3 we present more carefully the notion of good reduction.

The first author studied some problems linked to Uniform Boundedness Conjecture. In particular, when $N = 1$, K is a number field and $\phi: \mathbb{P}_1 \rightarrow \mathbb{P}_1$ is an endomorphism defined over K , he proved in [3, Theorem 1] that the length of a cycle of a preperiodic point of ϕ is bounded by a number depending only on the cardinality of the set of places of bad reduction of ϕ .

A similar result in the function field case was recently proved in [4]. Furthermore in the same paper there is a bound proved for number fields, that is slightly better than the one in [3].

Theorem 1.1 (Theorem 1, [4]). *Let K be a global field. Let S be a finite set of places of K , containing all the archimedean ones, with cardinality $|S| \geq 1$. Let p be the characteristic of K . Let $D = [K : \mathbb{F}_p(t)]$ when $p > 0$, or $D = [K : \mathbb{Q}]$ when $p = 0$. Then there exists a number $\eta(p, D, |S|)$, depending only on p, D and $|S|$, such that if $P \in \mathbb{P}_1(K)$ is a preperiodic point for an endomorphism ϕ of \mathbb{P}_1 defined over K with good reduction outside S , then $|O_{\phi}(P)| \leq \eta(p, D, |S|)$. We can choose*

$$\eta(0, D, |S|) = \max \left\{ (2^{16|S|-8} + 3) [12|S| \log(5|S|)]^D, [12(|S| + 2) \log(5|S| + 5)]^{4D} \right\}$$

in zero characteristic and

$$(1) \quad \eta(p, D, |S|) = (p|S|)^{4D} \max \{ (p|S|)^{2D}, p^{4|S|-2} \}.$$

in positive characteristic.

Observe that the bounds in Theorem 1.1 do not depend on the degree d of ϕ . As a consequence of that result, we could give the following bound for the cardinality of the set of K -rational preperiodic points for an endomorphism ϕ of \mathbb{P}_1 defined over K .

Corollary 1.1.1 (Corollary 1.1, [4]). *Let K be a global field. Let S be a finite set of places of K of cardinality $|S|$ containing all the archimedean ones. Let p be the characteristic of K . Let D be the degree of K over the rational function field $\mathbb{F}_p(t)$, in the positive characteristic, and over \mathbb{Q} , in the zero characteristic. Let $d \geq 2$ be an integer. Then there exists a number $C = C(p, D, d, |S|)$, depending only on p, D, d and $|S|$, such that for any endomorphism ϕ of \mathbb{P}_1 of degree d , defined over K and with good reduction outside S , we have*

$$\#\text{PrePer}(\phi, \mathbb{P}_1(K)) \leq C(p, D, d, |S|).$$

Theorem 1.1 extends to global fields and to preperiodic points the result proved by Morton and Silverman in [7, Corollary B]. The condition $|S| \geq 1$ in its statement is only a technical one. In the case of number fields, we require that S contains the archimedean places (i.e. the ones at infinity), then it is clear that the cardinality of S is not zero. In the function field case any place is non archimedean. Recall that the place at infinity in the case $K = \mathbb{F}_p(t)$ is the one associated to the valuation given by the prime element $1/t$. When K is a finite extension of $\mathbb{F}_p(t)$, the places at infinity of K are the ones that extend the place of $\mathbb{F}_p(t)$ associated to $1/t$. The arguments used in the proof of Theorem 1.1 and Corollary 1.1.1 work also when S does not contain all the places at infinity. Anyway, the most important situation is when all the ones at infinity are in S . For example, in order to have that any polynomial in $\mathbb{F}_p(t)$ is an S -integer, we have to put in S all those places. Note that in the number field case the quantity $|S|$ depends also on the degree D of the extension K of \mathbb{Q} , because S contains all archimedean places (whose amount depends on D).

Even when the cardinality of S is small, the bounds in Theorem 1.1 is quite big. This is a consequence of our searching for some uniform bounds (depending only on $p, D, |S|$). The bound $C(p, D, d, |S|)$ in Corollary 1.1.1 can be effectively given, but in this case too the bound is big, even for small values of the parameters $p, D, d, |S|$. For a much smaller bound see for instance the one proved by Benedetto in [1] for the case where ϕ is a polynomial. In the more general case when ϕ is a rational function with good reduction outside a finite S , the bound in Theorem 1.1 can be significantly improved for some particular sets S . For example if $K = \mathbb{Q}$ and S contains only the place at infinity, then we have the following bounds (see [4]):

- If $P \in \mathbb{P}_1(\mathbb{Q})$ is a periodic point for ϕ with minimal period n , then $n \leq 3$.
- If $P \in \mathbb{P}_1(\mathbb{Q})$ is a preperiodic point for ϕ , then $|O_\phi(P)| \leq 12$.

Here we prove some analogous bounds when $K = \mathbb{F}_p(t)$.

Theorem 1.2. *Let $\phi: \mathbb{P}_1 \rightarrow \mathbb{P}_1$ of degree $d \geq 2$ defined over $\mathbb{F}_p(t)$ with good reduction at every finite place. If $P \in \mathbb{P}_1(\mathbb{F}_p(t))$ is a periodic point for ϕ with minimal period n , then*

- $n \leq 3$ if $p = 2$;
- $n \leq 72$ if $p = 3$
- $n \leq (p^2 - 1)p$ if $p \geq 5$.

More generally if $P \in \mathbb{P}_1(\mathbb{F}_p(t))$ is a preperiodic point for ϕ we have

- $|O_\phi(P)| \leq 9$ if $p = 2$;
- $|O_\phi(P)| \leq 288$ if $p = 3$;
- $|O_\phi(P)| \leq (p + 1)(p^2 - 1)p$ if $p \geq 5$.

Observe that the bounds do not depend on the degree of ϕ but they depend only on the characteristic p . In the proof we will use some ideas already written in [2], [3] and [4]. The original idea of using S -unit theorems in the context of the arithmetic of dynamical systems is due to Narkiewicz [9].

2. VALUATIONS, S -INTEGERS AND S -UNITS

We adopt the present notation: let K be a global field and v_p a valuation on K associated to a non archimedean place p . Let $R_p = \{x \in K \mid v_p(x) \geq 1\}$ be the local ring of K at p .

Recall that we can associate an absolute value to any valuation v_p , or more precisely a place p that is a class of absolute values (see [5] and [12] for a reference about this topic). With $K = \mathbb{F}_p(t)$, all places are exactly the ones associated either to a monic irreducible polynomial in $\mathbb{F}_p[t]$ or to the place at infinity given by the valuation $v_\infty(f(x)/g(x)) = \deg(g(x)) - \deg(f(x))$, that is the valuation associated to $1/t$.

In an arbitrary finite extension K of $\mathbb{F}_p(t)$ the valuations of K are the ones that extend the valuations of $\mathbb{F}_p(t)$. We shall call places at infinity the ones that extend the above valuation v_∞ on $\mathbb{F}_p(t)$. The other ones will be called finite places. The situation is similar to the one in number fields. The non archimedean places in \mathbb{Q} are the ones associated to the valuations at any prime p of \mathbb{Z} . But there is also a place that is not non-archimedean, the one associated to the usual absolute value on \mathbb{Q} . With an arbitrary number field K we call archimedean places all the ones that extend to K the place given by the absolute value on \mathbb{Q} .

From now on S will be a finite fixed set of places of K . We shall denote by

$$R_S := \{x \in K \mid v_p(x) \geq 0 \text{ for every prime } p \notin S\}$$

the ring of S -integers and by

$$R_S^* := \{x \in K^* \mid v_p(x) = 0 \text{ for every prime } p \notin S\}$$

the group of S -units.

As usual let $\overline{\mathbb{F}}_p$ be the algebraic closure of \mathbb{F}_p . The case when $S = \emptyset$ is trivial because if so, then the ring of S -integers is already finite; more precisely $R_S = R_S^* = K^* \cap \overline{\mathbb{F}}_p$. Therefore in what follows we consider $S \neq \emptyset$.

In any case we have that $K^* \cap \overline{\mathbb{F}}_p$ is contained in R_S^* . Recall that the group $R_S^*/K^* \cap \overline{\mathbb{F}}_p$ has finite rank equal to $|S| - 1$ (e.g. see [10, Proposition 14.2 p.243]). Thus, since $K \cap \overline{\mathbb{F}}_p$ is a finite field, we have that R_S^* has rank equal to $|S|$.

3. GOOD REDUCTION

We shall state the notion of good reduction following the presentation given in [11] and in [4].

Definition 3.0.1. Let $\Phi : \mathbb{P}_1 \rightarrow \mathbb{P}_1$ be a rational map defined over K , of the form

$$\Phi([X : Y]) = [F(X, Y) : G(X, Y)]$$

where $F, G \in K[X, Y]$ are coprime homogeneous polynomials of the same degree. We say that Φ is in \mathfrak{p} -reduced form if the coefficients of F and G are in $R_{\mathfrak{p}}[X, Y]$ and at least one of them is a \mathfrak{p} -unit (i.e. a unit in $R_{\mathfrak{p}}$).

Recall that $R_{\mathfrak{p}}$ is a principal local ring. Hence, up to multiplying the polynomials F and G by a suitable non-zero element of K , we can always find a \mathfrak{p} -reduced form for each rational map. We may now give the following definition.

Definition 3.0.2. Let $\Phi : \mathbb{P}_1 \rightarrow \mathbb{P}_1$ be a rational map defined over K . Suppose that the morphism $\Phi([X : Y]) = [F(X, Y) : G(X, Y)]$ is written in \mathfrak{p} -reduced form. The *reduced map* $\Phi_{\mathfrak{p}} : \mathbb{P}_{1, k(\mathfrak{p})} \rightarrow \mathbb{P}_{1, k(\mathfrak{p})}$ is defined by $[F_{\mathfrak{p}}(X, Y) : G_{\mathfrak{p}}(X, Y)]$, where $F_{\mathfrak{p}}$ and $G_{\mathfrak{p}}$ are the polynomials obtained from F and G by reducing their coefficients modulo \mathfrak{p} .

With the above definitions we give the following one:

Definition 3.0.3. A rational map $\Phi : \mathbb{P}_1 \rightarrow \mathbb{P}_1$, defined over K , has *good reduction* at \mathfrak{p} if $\deg \Phi = \deg \Phi_{\mathfrak{p}}$. Otherwise we say that it has bad reduction at \mathfrak{p} . Given a set S of places of K containing all the archimedean ones. We say that Φ has good reduction outside S if it has good reduction at any place $\mathfrak{p} \notin S$.

Note that the above definition of good reduction is equivalent to ask that the homogeneous resultant of the polynomial F and G is invertible in $R_{\mathfrak{p}}$, where we are assuming that $\Phi([X : Y]) = [F(X, Y) : G(X, Y)]$ is written in \mathfrak{p} -reduced form.

4. DIVISIBILITY ARGUMENTS

We define the p -adic logarithmic distance as follows (see also [8]). The definition is independent of the choice of the homogeneous coordinates.

Definition 4.0.4. Let $P_1 = [x_1 : y_1], P_2 = [x_2 : y_2]$ be two distinct points in $\mathbb{P}_1(K)$. We denote by

$$(2) \quad \delta_p(P_1, P_2) = v_p(x_1 y_2 - x_2 y_1) - \min\{v_p(x_1), v_p(y_1)\} - \min\{v_p(x_2), v_p(y_2)\}$$

the p -adic logarithmic distance.

The divisibility arguments, that we shall use to produce the S -unit equation useful to prove our bounds, are obtained starting from the following two facts:

Proposition 4.0.1. [8, Proposition 5.1]

$$\delta_p(P_1, P_3) \geq \min\{\delta_p(P_1, P_2), \delta_p(P_2, P_3)\}$$

for all $P_1, P_2, P_3 \in \mathbb{P}_1(K)$.

Proposition 4.0.2. [8, Proposition 5.2] Let $\phi: \mathbb{P}_1 \rightarrow \mathbb{P}_1$ be a morphism defined over K with good reduction at a place p . Then for any $P, Q \in \mathbb{P}(K)$ we have

$$\delta_p(\phi(P), \phi(Q)) \geq \delta_p(P, Q).$$

As a direct application of the previous propositions we have the following one.

Proposition 4.0.3. [8, Proposition 6.1] Let $\phi: \mathbb{P}_1 \rightarrow \mathbb{P}_1$ be a morphism defined over K with good reduction at a place p . Let $P \in \mathbb{P}(K)$ be a periodic point for ϕ with minimal period n . Then

- $\delta_p(\phi^i(P), \phi^j(P)) = \delta_p(\phi^{i+k}(P), \phi^{j+k}(P))$ for every $i, j, k \in \mathbb{Z}$.
- Let $i, j \in \mathbb{N}$ such that $\gcd(i - j, n) = 1$. Then $\delta_p(\phi^i(P), \phi^j(P)) = \delta_p(\phi(P), P)$.

5. PROOF OF THEOREM 1.2

We first recall the following statements.

Theorem 5.1 (Morton and Silverman [8], Zieve [14]). Let K, p, p be as above. Let Φ be an endomorphism of \mathbb{P}_1 of degree at least two defined over K with good reduction at p . Let $P \in \mathbb{P}_1(K)$ be a periodic point for Φ with minimal period n . Let m be the primitive period of the reduction of P modulo p and r the multiplicative period of $(\Phi^m)'(P)$ in $k(p)^*$. Then one of the following three conditions holds

- (i) $n = m$;
- (ii) $n = mr$;
- (iii) $n = p^e mr$, for some $e \geq 1$.

In the notation of Theorem 5.1, if $(\Phi^m)'(P) = 0$ modulo p , then we set $r = \infty$. Thus, if P is a periodic point, then the cases (ii) and (iii) are not possible with $r = \infty$.

Proposition 5.1.1. [8, Proposition 5.2] Let $\phi: \mathbb{P}_1 \rightarrow \mathbb{P}_1$ be a morphism defined over K with good reduction at a place p . Then for any $P, Q \in \mathbb{P}(K)$ we have

$$\delta_p(\phi(P), \phi(Q)) \geq \delta_p(P, Q).$$

Lemma 5.1.1. Let

$$(3) \quad P = P_{-m+1} \mapsto P_{-m+2} \mapsto \dots \mapsto P_{-1} \mapsto P_0 = [0 : 1] \mapsto [0 : 1].$$

be an orbit for an endomorphism ϕ defined over K with good reduction outside S . For any a, b integers such that $0 < a < b \leq m - 1$ and $p \notin S$, it holds

- a) $\delta_p(P_{-b}, P_0) \leq \delta_p(P_{-a}, P_0)$;
- b) $\delta_p(P_{-b}, P_{-a}) = \delta_p(P_{-b}, P_0)$.

Proof a) It follows directly from Proposition 5.1.1.

b) By Proposition 4.0.1 and part a) we have

$$\delta_p(P_{-b}, P_{-a}) \geq \min\{\delta_p(P_{-b}, P_0), \delta_p(P_{-a}, P_0)\} = \delta_p(P_{-b}, P_0).$$

Let r be the largest positive integer such that $-b + r(b - a) < 0$. Then

$$\begin{aligned} \delta_p(P_{-b}, P_0) &\geq \min\{\delta_p(P_{-b}, P_{-a}), \delta_p(P_{-a}, P_{b-2a}), \dots, \delta_p(P_{-b+r(b-a)}, P_0)\} \\ &= \delta_p(P_{-b}, P_{-a}). \end{aligned}$$

The inequality is obtained by applying Proposition 4.0.1 several times. \square

Lemma 5.1.2 (Lemma 3.2 [4]). *Let K be a function field of degree D over $\mathbb{F}_p(t)$ and S a non empty finite set of places of K . Let $P_i \in \mathbb{P}_1(K)$ with $i \in \{0, \dots, n-1\}$ be n distinct points such that*

$$(4) \quad \delta_p(P_0, P_1) = \delta_p(P_i, P_j), \text{ for each distinct } 0 \leq i, j \leq n-1 \text{ and for each } p \notin S.$$

Then $n \leq (p|S|)^{2D}$.

Since $\mathbb{F}_p(t)$ is a principal ideal domain, every point in $\mathbb{P}_1(\mathbb{F}_p(t))$ can be written in S -coprime coordinates, i. e., for each $P \in \mathbb{P}_1(\mathbb{F}_p(t))$ we may write $P = [a : b]$ with $a, b \in R_S$ and $\min\{v_p(a), v_p(b)\} = 0$, for each $p \notin S$. We say that $[a : b]$ are S -coprime coordinates for P .

Proof of Theorem 1.2 We use the same notation of Theorem 5.1. Assume that S contains only the place at infinity. *Case $p = 2$.* Let $P \in \mathbb{P}_1(\mathbb{F}_p(t))$ be a periodic point for ϕ . Without loss of generality we can suppose that $P = [0 : 1]$. Observe that m is bounded by 3 and $r = 1$. By Theorem 5.1, we have $n = m \cdot 2^e$, with e a non negative integral number. Up to considering the m -th iterate of ϕ , we may assume that the minimal periodicity of P is 2^e . So now suppose that $n = 2^e$, with $e \geq 2$. Consider the following 4 points of the cycle:

$$[0 : 1] \mapsto [x_1 : y_1] \mapsto [x_2 : y_2] \mapsto [x_3 : y_3] \dots$$

where the points $[x_i : y_i]$ are written S -coprime integral coordinates for all $i \in \{1, \dots, n-1\}$. By applying Proposition 4.0.3, we have $\delta_p([0 : 1], P_1) = \delta_p([0 : 1], P_3)$, i. e. $x_3 = x_1$, because of $R_S^* = \{1\}$. From $\delta_p([0 : 1], P_1) = \delta_p(P_1, P_2)$ we deduce

$$(5) \quad y_2 = \frac{x_2}{x_1} y_1 + 1.$$

Furthermore, by Proposition 4.0.3 we have $\delta_p([0 : 1], P_1) = \delta_p(P_2, P_3)$. Since $x_3 = x_1$, then

$$(6) \quad y_3 x_2 - x_3 y_2 = x_1.$$

This last equality combined with (5) provides $y_3 = y_1$, implying $[x_1 : y_1] = [x_3 : y_3]$. Thus $e \leq 1$ and $n \in \{1, 2, 3, 6\}$. The next step is to prove that $n \neq 6$. If $n = 6$, with few calculations one sees that the cycle has the following form.

$$(7) \quad [0 : 1] \mapsto [x_1 : y_1] \mapsto [A_2 x_1 : y_2] \mapsto [A_3 x_1 : y_3] \mapsto [A_2 x_1 : y_4] \mapsto [x_1 : y_5] \mapsto [0 : 1],$$

where $A_2, A_3 \in R_S$ and everything is written in S -coprime integral coordinates. We may apply Proposition 4.0.3, then by considering the p -adic distances $\delta_p(P_1, P_i)$ for all indexes $2 \leq i \leq 5$ for every place p , we obtain that there exists some S -units u_i such that

$$(8) \quad y_2 = A_2 y_1 + u_2; \quad y_3 = A_3 y_1 + A_2 u_3; \quad y_4 = A_2 y_1 + A_3 u_4; \quad y_5 = y_1 + A_2 u_5.$$

Since $R_S^* = \{1\}$, we have that the identities in (8) become

$$y_2 = A_2 y_1 + 1; \quad y_3 = A_3 y_1 + A_2; \quad y_4 = A_2 y_1 + A_3; \quad y_5 = y_1 + A_2$$

where A_2, A_3 are non zero elements in $\mathbb{F}_p[t]$. By considering the p -adic distance $\delta_p(P_2, P_4)$ for each finite place p , from Proposition 4.0.3 we obtain that

$$v_p(A_2 x_1) = \delta_p(P_2, P_4) = v_p(A_2 x_1 (A_2 y_1 + A_3) - A_2 x_1 (A_2 y_1 + 1)) = v_p(A_2 A_3 x_1 - A_2 x_1),$$

i. e. $A_2 x_1 = A_2 A_3 x_1 - A_2 x_1$ (because $R_S^* = \{1\}$). Then $A_2 A_3 x_1 = 0$ that contradicts $n = 6$. Thus $n \leq 3$.

Suppose now that P is a preperiodic point. Without loss of generalities we can assume that the orbit of P has the following shape:

$$(9) \quad P = P_{-m+1} \mapsto P_{-m+2} \mapsto \dots \mapsto P_{-1} \mapsto P_0 = [0 : 1] \mapsto [0 : 1].$$

Indeed it is sufficient to take in consideration a suitable iterate ϕ^n (with $n \geq 3$), so that the orbit of the point P , with respect the iterate ϕ^n , contains a fixed point P_0 . By a suitable conjugation by an element of $\text{PGL}_2(R_S)$, we may assume that $P_0 = [0 : 1]$.

For all $1 \leq j \leq m-1$, let $P_{-j} = [x_j : y_j]$ be written in S -coprime integral coordinates. By Lemma 5.1.1, for every $1 \leq i < j \leq m-1$ there exists $T_{i,j} \in R_S$ such that $x_i = T_{i,j} x_j$. Consider the p -adic distance between the points P_{-1} and P_{-j} . Again by Lemma 5.1.1, we have

$$\delta_p(P_{-1}, P_{-j}) = v_p(x_1 y_j - x_1 y_1 / T_{1,j}) = v_p(x_1 / T_{1,j}),$$

for all $p \notin S$. Then, there exists $u_j \in R_S^*$ such that $y_j = (y_1 + u_j) / T_{1,j}$, for all $p \notin S$. Thus, there exists $u_j \in R_S^*$ such that $[x_{-j}, y_{-j}] = [x_1, y_1 + u_j]$. Since $R_S^* = \{1\}$, then $P_{-j} = [x_1 : y_1 + 1]$. So the length of the orbit (9) is at most 3. We get the bound 9 for the cardinality of the orbit of P .

Case $p > 2$.

Since $D = 1$ and $|S| = 1$, then the bound for the number of consecutive points as in Lemma 5.1.2 can be chosen equal to p^2 . By Theorem 5.1 the minimal periodicity n for a periodic point $P \in \mathbb{P}_1(\mathbb{Q})$ for the map ϕ is of the form $n = m r p^e$ where $m \leq p+1$, $r \leq p-1$ and e is a non negative integer.

Let us assume that $e \geq 2$. Since $p > 2$, by Proposition 4.0.3, for every $k \in \{0, 1, 2, \dots, p^{e-2}\}$ and $i \in \{2, \dots, p-1\}$, we have that $\delta_p(P_0, P_1) = \delta_p(P_0, P_{k \cdot p + i})$, for any $p \notin S$. Then $P_{k \cdot p + i} = [x_1, y_{k \cdot p + i}]$. Furthermore $\delta_p(P_0, P_1) = \delta_p(P_0, P_{k \cdot p + i})$ implying that there exists a element $u_{k \cdot p + i} \in R_S^*$ such that

$$(10) \quad P_{k \cdot p + i} = [x_1 : y_1 + u_{k \cdot p + i}].$$

Since R_S^* has $p-1$ elements and there are $(p^{e-2}+1)(p-2)$ numbers of the shape $k \cdot p + i$ as above, we have $(p^{e-2}+1)(p-2) \leq p-1$. Thus $e = 2$ and $p = 3$.

Then $n \leq 72$ if $p = 3$ and $n \leq (p^2-1)p$ if $p \geq 5$. For the more general case when P is preperiodic, consider the same arguments used in the case when $p = 2$, showing $[x_{-j}, y_{-j}] = [x_1, y_1 + u_j]$, with $u_j \in R_S^*$. Thus, the orbit of a point $P \in \mathbb{P}_1(\mathbb{Q})$ containing $P_0 \in \mathbb{P}_1(\mathbb{Q})$, as in (9), has length at most $|R_S^*| + 2 = p + 1$. The bound in the preperiodic case is then 288 for $p = 3$ and $(p+1)(p^2-1)p$ for $p \geq 5$. \square

With similar proofs, we can get analogous bounds for every finite extension K of $\mathbb{F}_p(t)$. The bounds of Theorem 1.2, with $K = \mathbb{F}_p(t)$, are especially interesting, for they are very small.

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